Lecture 8

Thursday, 27 September 2024   10:24 PM

EFX: $\forall$ agents N, $\forall$ goods M, v_{ig} is agent i's value for good g.

An allocation A is EFX if $\forall i,j \in N$:
$v_i(A_i) < v_i(A_j) \implies \forall g \in A_j,\ v_i(A_i) \geq v_i(A_j \setminus g)$

Say agent i strongly envies agent j if this is violated.

Then A is an EFX allocation iff there is no strong envy.

Open Q: For additive goods, does $\exists$ an EFX allocation? (settled recently for additive chores, & subadditive goods, negatively)

① Identical valuations:
Assume that $\forall i,j \in N$, $\forall S \subseteq M$, $v_i(S) = v_j(S) = v(S)$.
For allocation A, define $\vec{v}(A) = (v(A_1), \dots, v(A_n))_{\text{sorted}}$ sorted in increasing order.

Say $\vec{v}(A) > \vec{v}(B)$ if for some $i \in [n]$, $(\vec{v}(A))_i > (\vec{v}(B))_i$ & $\forall j < i$, $(\vec{v}(A))_j = (\vec{v}(B))_j$.

Thus, $\vec{v}(A) > \vec{v}(B)$ if either least value in $v(A) >$ least value in $v(B)$
or these are equal & 2nd least value in $v(A) >$ 2nd least value in $v(B)$
& both of these are equal & ...

Note that for any 2 allocations A, A',
either $\vec{v}(A) > \vec{v}(A')$ or $\vec{v}(A') > \vec{v}(A)$ (or $A = A'$).
i.e., this imposes a total order on the set of allocations.
This is called a lexicographic ordering.
Let A^* be the allocation s.t. $\vec{v}(A^*) \geq \vec{v}(A)$ $\forall A \neq A^*$.
A^* is called the leximin (or lexicographic minimum) allocation.

Theorem: For identical monotone valuations, the leximin allocation is EFX.

Proof: Let A^* be the leximin allocation, & for a contradiction assume A^* not EFX. Order agents so that
$v(A_1) \leq v(A_2) \leq \dots \leq v(A_n)$.
Then $\exists\, i < j$ s.t. $\exists g \in A_j$: $v(A_i) < v(A_j \setminus g)$.
Now consider the allocation A' where $A'_i = A_i \cup g$, $A'_j = A_j \setminus g$.
Note that no smaller value of $v(A)$ has changed.
Then $v(A'_i) > v(A_i)$ & $v(A'_j) > v(A_i)$,
$\forall k \neq i,j$: $v(A'_k) = v(A_k)$.
Thus $\vec{v}(A') > \vec{v}(A^*)$, & A^* cannot be the leximin allocation. ∎

Theorem: For 2 agents with identical monotone non-decreasing submodular valuations:
① Finding an EFX allocation has exponential query complexity, and
② is PLS-complete.

Theorem: For 2 agents with (non-identical) monotone non-decreasing valuations, $\exists$ an EFX allocation.

Proof: Cut & choose.
Assume 2 identical agents, both with valuation v_1.
Let $A = (A_1, A_2)$ be an EFX allocation. (agent 1 cuts)
Note that no matter which of the two bundles agent 2 gets, she does not strongly envy agent 2.
Agent 2 chooses her preferred bundle, agent 1 gets the remaining bundle.

Theorem: For 3 types of agents of additive monotone valuations, $\exists$ an EFX allocation.
Agents are of the same type if they have the same valuation.

For n agents of additive valuations, the problem is open.

Identical ordering: An instance I is identically ordered if
$\exists \sigma: [m] \to [m]$ s.t. $\forall$ agents i,
$v_{i\sigma(1)} \geq v_{i\sigma(2)} \geq \dots \geq v_{i\sigma(m)}$.

Consider the envy cycle elimination algorithm where items are allocated in this order.

Theorem: The algorithm returns an EFX allocation.
(Prove yourself)

Claim: Let A be an EFX (partial) allocation, G be the envy graph.
Let C be a directed cycle in the envy graph, & A' be obtained by resolving C. Then A' is EFX.

Proof: Note that the multiset of bundles has not changed.
For any agent i outside C, the multiset of bundles and her own bundle has not changed.
Let $\{B_1, \dots, B_n\}$ be the multiset of bundles.
Then if $v_i(A_i) < v_i(B_k)$, then $\forall g \in B_k$, $v_i(A_i) \geq v_i(B_k \setminus g)$
& $v_i(A'_i) = v_i(A_i)$ & this still holds.
Similarly for an agent in the cycle, the multiset of bundles has not changed, & her own value has strictly improved. ∎

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2 types of relaxations:

① An allocation is α-EFX if $\forall i,j$,
$v_i(A_i) < v_i(A_j) \implies \forall g \in A_j,\ v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus g)$

(ii) Envy with charity:
Let $A = (A_1, \dots, A_n)$ be a partial allocation, & $C = G \setminus \bigcup_i A_i$.
Then A is an EFX allocation with charity if:
- A is EFX,
- no agent envies C.

Approximate EFX

A valuation fn. $v: 2^M \to \mathbb{Z}_+$ is subadditive if $\forall S, T \subseteq G$,
$v(S) + v(T) \geq v(S \cup T)$

Theorem: For agents w/ subadditive valuations $\exists$ a $1/2$-EFX allocation.

> **Algorithm:**
> Let $A_i = \phi$ $\forall i \in N$, $R = M$
> While $R \neq \phi$
> Resolve any envy cycles
> If $\exists i \in N$, $g \in R$ s.t. $v_i(g) > v_i(A_i)$
> $R \leftarrow R \cup A_i$, $A_i \leftarrow g$ (swap A_i w/ g)
> Else
> give an arbitrary good from R to a source in the graph.

Proof: By induction.
Initially, A is $1/2$-EFX
Resolving envy cycles maintains $1/2$-EFX.
In the swap operation, where agent i swaps:
- no agent strongly envies agent i
- agent i's value strictly improves, & the multiset of bundles she sees is unchanged.

If a good g is allocated to a source i:
- i's value goes up, & the multiset of bundles she sees is unchanged
- Let A be allocation before allocating good g. Then $\forall j \neq i$
$v_j(A_i \cup g) \leq v_j(A_i) + v_j(g) \leq v_j(A_j) + v_j(A_j) = 2 v_j(A_j)$
(first inequality: since i was a source; second: since j did not swap her bundle for g) ∎

EFX with Charity

Theorem: For any instance w/ monotone valuations, there exists a partial allocation $A = (A_1, \dots, A_n)$ w/ charity $C = M \setminus \bigcup_i A_i$ s.t.
① A is EFX
② $\forall i \in N$, $v_i(A_i) \geq v_i(C)$
③ $|C| \leq n-1$

Defn.: Let A be a partial allocation, B be a bundle of goods $B \subseteq M$.
Agent i is a most envious agent of B if:
1. $\exists S \subseteq B$ s.t. $v_i(A_i) < v_i(S)$, and
2. $\forall S' \subsetneq S$, no agent envies S' (including i)
i.e. $\forall j \in N$, $v_j(A_j) \geq v_j(S')$

The set S is called the minimal envied subset.

Now if we just want ① & ②, this is easy.
Algo:
$\forall i$ $A_i = \phi$, $C = M$
While any agent envies C:
Let i be a most envious agent of C, S be a minimal envied subset of C.
$C \leftarrow C \cup A_i$, $A_i \leftarrow S$

Claim: The algo returns an EFX partial allocation where no agent envies C.

Proof: The algorithm clearly terminates, since in every iteration, some agent's value strictly increases.
If A is EFX before an iteration, then it is EFX after:
i's value strictly increases while other bundles are unchanged, & no agent strictly envies i.

Given a partial allocation A, agent i, let $P(i)$ be the set of agents that i has a directed path to in the envy graph.

Algo for ① ② & ③:

Rule 0: Resolve any envy cycles

If rule 0 not applicable:

Rule 1: If $\exists g \in C$ & $i \in N$ s.t. allocating g to s results in an EFX allocation:
- give g to s
(note that s must be a source in the envy graph, & minimal envied set of $A_i \cup s$ is $A_i \cup s$.)

If rules 0 & 1 not applicable:

Rule 2: If $\exists i \in N$ s.t. $v_i(C) > v_i(A_i)$:
- Let i be a most envious agent of C, $S \subseteq C$ be a minimal envied subset.
- Exchange A_i & S

If rules 0, 1, & 2 are not applicable:

Rule 3: Find a set of k distinct goods $g_1, \dots, g_k \in C$, k distinct sources $s_1, \dots, s_k$, & k distinct agents $t_1, \dots, t_k$ so that:
$\forall i \in [k]$, t_i is a most envious agent of $A_{s_i} \cup g_i$,
$t_i \notin \bigcup_{j \leq i} P(s_j)$, &
$t_i \in P(s_{i+1})$
- Let $Z_i \subseteq A_{s_i} \cup g_i$ be the minimal envied subset of $A_{s_i} \cup g_i$
- Let P_i be the path from s_i to t_{i-1}.

Claim: The paths $P_1, \dots, P_k$ are all distinct.

Proof: Say $v \in P_i \cap P_j$, & $i < j$
P_i is a path from s_i to t_{i-1}, P_j: $s_j \to t_{j-1}$.
But then $t_{j-1} \in P(s_i)$ for $j > i$, which is not allowed. ∎

- Construct allocation A as follows:
$\forall i \in [k]$:
$A_{t_i} \leftarrow Z_i$, $C \leftarrow C \cup A_{t_i}$
every agent in P_i except t_{i-1} gets the successive agent's bundle.

Claim: If A is EFX, then after application of rule 3, the resulting allocation is EFX.

Proof: Firstly, for every agent in $\bigcup_{i \in [k]} P_i$, the value strictly improves. Other agents are unchanged.
Secondly, the multiset of bundles is unchanged, except every bundle A_{t_i} is replaced by Z_i. But Z_i is a minimal envied subset, so no agent envies any strict subset of Z_i. Hence EFX.

Claim: If $|C| \geq n$ & rules 0, 1, 2 do not hold, then rule 3 must hold.
(prove yourself)